Technological innovation and new mathematics: van der Pol and the birth of non-linear dynamics[(*)]

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1. Introduction

It is sometimes claimed that the emergence of a form of mathematization of phenomena based on the use of non-linear mathematical models resulted from (or was at least favoured by) the needs of 1940s technology, in particular during World War II. There is no doubt that the introduction of a large number of new applications or indeed of new branches of mathematics was a response to the wartime situation. The period that began in the '40s saw the development of game theory, linear and non-linear programming, cybernetics, the science of digital calculus, information theory, and non-linear dynamics. It is equally well known that the notions of feed-back and servomechanism played a central role in these developments, which thus seem to be closely related to a profound change in technological conceptions. It would however be somewhat superficial to overlook the fact that the roots of these developments originate in the earlier past, in particular in the case of non-linear modelling and the analysis of feed-back processes.

The search for the "precursors" in historical analysis leads to superficial relations being established and to the specific nature of the different contexts being concealed. The truth contained in such slogans as "Democritus is the father of modern atomic theory" is not enough to hide its sterility. In our particular case, it is not very relevant to recall to mind a few of Maxwell's intuitive thoughts to anchor the origins of the modern analysis of servomechanisms. Nor is it relevant to revisit the work of Claude Bernard in order to introduce the concept of "homeostasis" (which represents the very core of the notion of feed-back and thus much of non-linear dynamics). It is true that, in both cases, we are dealing with important anticipations: however, they are linked to a conceptual and practical context that is different and remote. Nevertheless, even sticking to the "modern" form in which these ideas were explicitly expressed and disseminated during the second half of the 20th century, there seems to be no doubt that they were introduced during its early decades and not in the '40s or '50s. To fix the origin of the idea of homeostasis, rather than cite Wiener, it is necessary to revisit the work of W. B. Cannon, which takes us back to the beginning of the 'thirties and to topics developed in the area of human physiology. Another example is provided by the origins of such an important theory in non-linear dynamics as "Hopf's bifurcation". Hopf's theorem dates to 1942, although the technological reasons underlying the theory may be sought in the study of the properties (and anomalies) of the behaviour of certain servomechanisms used on steam engines at the beginning of the century. And yet it is only in the context of interest in the more advanced technologies such as radio engineering, which we will illustrate here, and of the consequent organic development of the theory of non-linear oscillations, that this theory will be

[(*)] This paper (here in a preprint version) will be published in the book *Technological Concepts and Mathematical Models in the Evolution of Modern Engineering Systems* (M. Lucertini, A. Millán Gasca, F. Nicolò, eds.), forthcoming.

fully developed.

More generally, the theory of non-linear oscillations and the concept of limit cycle derive from the developments in late nineteenth century mathematical physics, and in particular in Henri Poincaré's celestial mechanics studies, as gathered together in the well-known treatise *Les méthodes nouvelles de la mécanique céleste*[1]. Nevertheless, the discovery and study of limit cycles by the Dutch engineer Balthazar L. van der Pol, in the '20s, took place in a completely different context and was closely linked to the emergence of new technologies. It was the study of the propagation of radio waves and of the electrical devices required to generate them that led van der Pol to work out the equation that is today considered as the prototype of the non-linear feedback oscillator. Furthermore, van der Pol's engineering work covers such a wide area in the study of a large number of devices of great relevance to modern technology (radio, telephone, colour TV) as to highlight the continuity linking this type of research to the developments that occurred in the second half of last century.

Russian mathematical physics was strongly influenced by the theoretical work of Poincaré and Lyapunoff and was sensitive to their links with engineering topics: the concept of limit cycle – which emerged also in the topics related to Hopf's bifurcation – was familiar to it. Conversely, van der Pol showed no awareness – at least in the work he did in the '20s – of Poincaré and Lyapunoff's work and does not even use the term "limit cycle" to define the regime of self-oscillating electrical circuits. He uses direct methods of analysis that have no relation to what was already known in the field of the qualitative theory of ordinary differential equations. He was reproached for the "primitive" nature of the mathematical methods he used by the Soviet school, who resumed his research and subjected it to a rigorous general re-elaboration, ultimately laying the foundations for a new "non-linear mechanics". Nevertheless, the impulse given by van der Pol's work was decisive in defining and directing the research of the Soviet school in the field of non-linear analysis. His work proposed completely new themes and drew upon a technological context that called much more strongly and directly for the elaboration of a general theory of non-linear oscillations (with feedback) than the context of steam engine technology. Van der Pol makes use of a rather old mathematical tool, but moves on the crest of the advanced technology wave of the time, which accompanies most of the developments occurring in the century. The Soviet researchers completely dominated the early developments of non-linear analysis, but related them to a classical context (celestial mechanics) as well as to technological forms of the previous century. Therefore, a fundamental turning point was reached through the synergy between the new ideas expressed in van der Pol's research and the advanced mathematical techniques developed by Mandelstam, Papalexi, Andronow, Witt, Kryloff and Bogoliuboff in their studies on non-linear self-oscillating systems.

However, there is another aspect in which van der Pol's contribution seems to be of great importance. As we have shown in other works[2], the views held by van der Pol on the mathematization of phenomena represent one of the first and most precise expressions of the modern conception of mathematical modelling. Its characteristics may be summed up in the words of John von Neumann: «… the sciences do not try to explain, they hardly even try to interpret, they mainly make models. By a model is meant a mathematical construct which, with the addition of certain verbal interpretations, describes observed phenomena. The justification of such a mathematical construct is solely and precisely that it is expected to work — that is, correctly to describe phenomena from a reasonably wide area»[3].

Our present aim is to illustrate these two profoundly innovative aspects of the scientific work

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1 Poincaré H. 1892-1899.
2 Israel G. 1996, 1998.
3 Neumann J. (von) 1955.

of Balthazar van der Pol.

2. From radio to limit cycles

Balthazar van der Pol was born on 27 January 1899 at Utrecht (Holland) where he did his university studies, graduating in physics in 1916. That same year he moved to England to complete his training in the radioelectricity laboratory directed by J. A. Fleming at University College, London. In 1917, he moved to the Cavendish Laboratory, which was directed at the time by J. J. Thomson, where he was introduced to the study of radio waves. He returned to Holland in 1919 and was awarded his Ph.D. from the University of Utrecht for a thesis on "The effect of an ionised gas on electro-magnetic wave propagation and its application to radio, as demonstrated by glow-discharge measurement". From 1919 to 1922 he worked as an assistant to H. A. Lorentz, at the Teyler Institute of Haarlem. In 1922 he joined the Philips company as head physicist in the research laboratory, before becoming director of the radio scientific research section. He stayed with Philips until 1949.

In the meantime, he also carried on academic activities as professor of theoretical electricity at the Technological University of Delft, from 1938 to 1949; in the years 1945 and 1946 he was president of the Temporary University of Eindhoven. In 1934 he was appointed vice-president of the Institute of Radio Engineers (USA), of which he had been a member since 1920, and he was awarded the Medal of Honour of this association for his contributions to the theory of electrical circuits. From 1934 to 1952 he was vice-president of the Union Radio Scientifique Internationale (U.R.S.I.) of which he was honorary president from 1952 on, as well as representative (1952 to 1959) on the Executive Board of the International Council of Scientific Unions. From 1949 to 1956 he directed the Comité Consultatif International des Radiocommunications (C.C.I.R.) in Geneva and was technical advisor to the International Telecommunications Union for the planning and development of radio communications in the early postwar period. After his retirement in 1956, he was invited to hold a chair for one year at the University of California in Berkeley and then at Cornell University. He died on 6 October 1959 at Wassenaar (Holland).

Van der Pol's technological and scientific interests took definitive shape during his stay at Cambridge under the guidance of J. J. Thomson. The great success achieved by Guglielmo Marconi in 1901, with his radio transmission across the Atlantic, had opened up a broad field of theoretical research aimed at accounting for the nature of the phenomenon. The most important contribution was Heaviside's hypothesis concerning the existence of an ionized atmospheric layer (the "ionosphere" or "Heaviside layer") which played a role in deviating radio waves. It should be noted that the actual existence of the ionosphere was discovered only in 1925, and at that time the concept was referred to only in hypothetical terms. One of van der Pol's first research topics was precisely the investigation of the hypothesis of ionic refraction, which would allow the transmission of radio waves around the Earth. In one of his early works[4] he showed that, without the hypothesis of an ionized layer, there would have been serious disagreement between theory and experimental evidence. The direct proof of the existence of ionic refraction required proof of the fact that the index of refraction of the ionized medium was less than 1. Starting from the principle that air ionized by an electric discharge produces a phenomenon of refraction of radio waves, he invented an apparatus that allowed him to compute the dielectric constant of an ionized gas by means of an electric discharge and to show that it could be varied and become less than 1[5]. In carrying out this experiment he used a triode oscillator that enabled him to produce waves with a wavelength of

[4] Pol B. L. (van der) 1919-1920.
[5] Pol B. L. (van der) 1919a, b.

about 3 metres, the shortest ever achieved so far. It is significant that this ability of van der Pol paved the way to the exploit that, in 1925, using a 200 kW triode transmitter, enabled him to establish radiotelephone contact between Holland and the Dutch East Indies, and for which he was decorated with the Order of Orange-Nassau.

What we have described above thus shows how van der Pol's research at Cambridge paved the wave for the theoretical analysis of radio wave propagation in the atmosphere (through the experimental investigation of their propagation in gases) and for the theory of electron motion in triodes. This research, carried out partly in collaboration with Edward Appleton, led to a series of publications written between 1920 and 1922 mainly on the subject of oscillation hysteresis and forced vibrations in a non-linear system[6]. Of particular importance in our case is the 1922 publication in collaboration with Appleton, as it contained an embryonic form of the equation of the triode oscillator, now referred to as "van der Pol's equation".

The authors examine a circuit of the type shown in Fig. 1A and point out that, in general, the anode current of a triode is a function of both the anode and the grid potential with respect to the filament, although here there exists a fixed relation between the variable parts v_a and v_g of the anode and the grid potentials, that may be expressed in the form: $v_g = -\frac{M}{L} v_a$, in which it is assumed that the grid currents can be neglected. The authors observe that: «... in such a case the variable part (i_a) of the anode current may be expressed as a function of v_a only, and it is precisely this relation which is represented by the oscillation characteristics. [...] In this way we are able to leave out the account of the retroactive action of the control electrode, and deal simply with the problem of a conductor possessing a characteristic relation $i_a = \psi(v_a)$ connected to an oscillatory circuit as shown in fig 1B.»

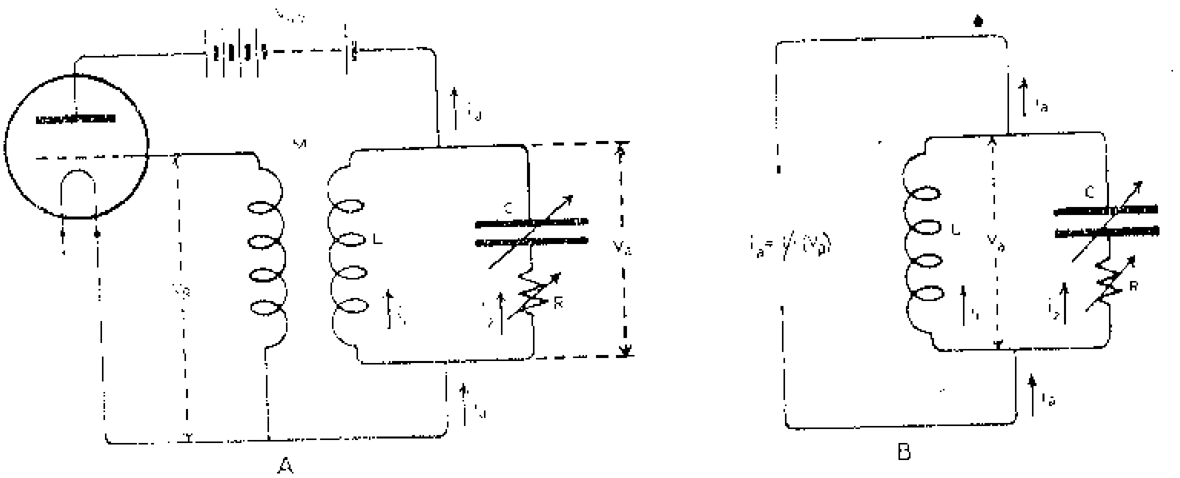

Fig. 1

The application of Kirchoff's law to this circuit leads to the following equations:

$$L\frac{di_1}{dt} = Ri_2 + \frac{1}{C}\int i_2 dt = -v_a$$

$$i_1 + i_2 = i_a = \psi(v_a)$$

from which can be derived the differential equation:

$$\frac{d^2}{dt^2}(v_a + Ri_a) + \frac{d}{dt}\left\{\frac{R}{L}v_a + \frac{\psi(v_a)}{C}\right\} + \frac{v_a}{CL} = 0 \tag{1}$$

[6] Pol B. L. (van der) 1920, 1922, Appleton E. V., Pol B. L. (van der) 1921, 1922.

In the case of a high frequency circuit, Ri_a is small compared with v_a and so the equation may be rewritten in the following form:

(2)

where it is assumed that $v_a = v$, $\omega_0^2 = \dfrac{1}{CL}$ and $\chi(v) = \left(\dfrac{Rv}{L} + \dfrac{\psi(v)}{C}\right)$.

This represents the first form of van der Pol's equation. Analysis of the equation leads to several partial results and to the determination of the characteristic form of the variable resistance, although it does not clearly indicate the presence of the limit cycle. A decisive step forward in this direction is however contained in the paper "On "Relaxation-Oscillations"" which van der Pol wrote on his own and published in 1926[7]. Here the author shows he now has very clear ideas concerning the nature of the process represented by the equation and about the form of the solutions.

Van der Pol considers the general form of an oscillatory system subjected to a dissipative force, namely $\ddot{x} + \alpha\dot{x} + \omega^2 x = 0$ and points out that, in the case in which the resistance is negative – as happens in certain electrical circuits (for instance, that of the triode) in which an energy input occurs – the equation becomes $\ddot{x} - \alpha\dot{x} + \omega^2 x = 0$. The solution of this second equation is however «physically unrealizable because it indicates an amplitude increasing to infinity. Thus for actual physical systems the differential equation will only be valid for values of x up to a certain value. To express the limitation of the amplitude we must assume that the coefficient of the "resistance" term is a function of the amplitude itself becoming positive at the higher values». In order to represent a situation of this kind, α is replaced with the expression $\alpha - 3\gamma x^2$, where γ is a constant. In this way the equation $\ddot{x} - (\alpha - 3\gamma x^2)\dot{x} + \omega^2 x = 0$ is obtained. In the case of an electrical circuit of the RLC (resistance-inductance-capacitance) type, such as a triode, $\alpha = \dfrac{R}{L}$ and $\omega^2 = \dfrac{1}{LC}$. By suitably changing the unit of measure of x and of time t, the equation can be rewritten in the following form (which is the one now customarily used for the van der Pol equation):

(3) $$\ddot{x} - \varepsilon(1 - x^2)\dot{x} + x = 0 \qquad \text{where } \varepsilon = \frac{\alpha}{\omega}.$$

Van der Pol considers this equation as a perturbation of the oscillator with negative resistance. When x is very small, equation (3) is reduced to the form of such an oscillator, $\ddot{x} - \varepsilon\dot{x} + x = 0$; it thus presents a solution tending towards infinity and is aperiodic. When the amplitude x increases, the non-linear term εx^2 cannot be neglected and, if $x^2 > 1$, it makes the second term of the equation positive by setting up a positive resistance and reducing the amplitude of the oscillations. Van der Pol does not provide a general proof that the presence of the non-linear term εx^2, *in any case*, regardless of the value of ε, provided that it is positive, determines a periodic solution towards which all the others tend asymptotically. He separately addresses the different numerical cases for ε, distinguishing the case in which $\varepsilon << 1$ from that in which $\varepsilon >> 1$. The first case corresponds to typical triode oscillations, in which a large number of periods are required to obtain a stationary state. An approximate solution of the equation shows that it tends definitively towards a periodic stationary state. Van der Pol declares that he expects a periodic stationary case to exist also in the case in which $\varepsilon >> 1$. Indeed, «although for small amplitudes the resistance has such a big negative value that the linear case would be highly aperiodic, the non-linear term, i.e. $x^2\dot{x}$, makes the solution periodic. We may thus say that we are dealing with a *quasi-aperiodic* solution.» Therefore, the case in which $\varepsilon >> 1$ can be distinguished from the others in that it is "quasi-aperiodic".

The explicit appearance of the limit cycle thus actually derives from the determination of

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[7] Pol B. L. (van der) 1926.

what today we call the phase diagram of the equation, which is obtained using the graphical isocline method in three distinct numerical cases: $\varepsilon = 0.1$, $\varepsilon = 1$, $\varepsilon = 10$. This is perhaps the most interesting aspect of van der Pol's work, in which, even above and beyond the analytical domain of the question, there appears a complete image of the non-linear oscillator named after him.

Let us rapidly examine these three cases and reproduce the author's original graphs. When $\varepsilon = 0.1$ (Fig. 2) a quasi sinusoidal oscillation of gradually increasing amplitude is obtained, the value of which ultimately becomes definitively stationary at a value of 2.

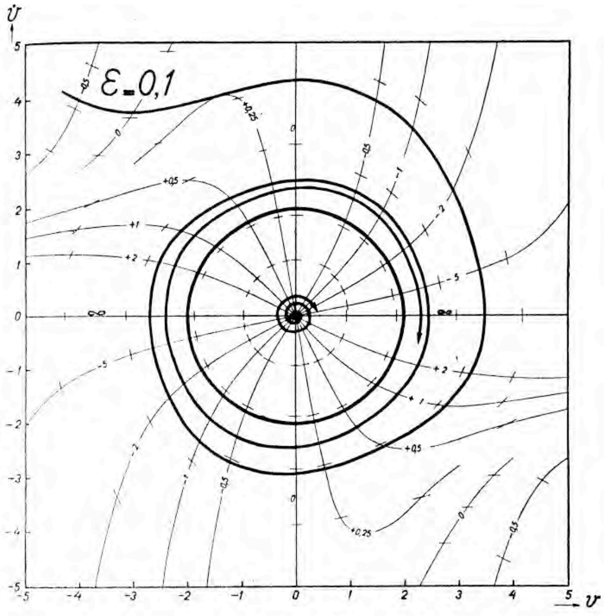

Fig. 2

The case in which $\varepsilon = 1$ «indicates a somewhat similar sequence of events, but here the final amplitude is reached in fewer oscillations, while a marked departure from the sinusoidal form is noticed».

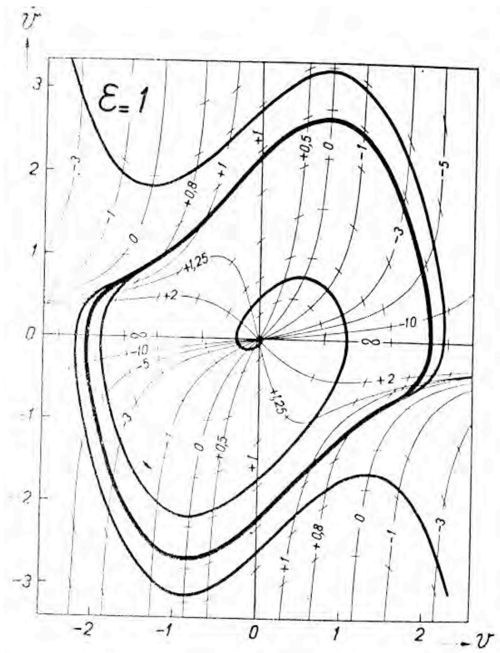

Fig. 3

A particularly interesting case is the one in which ε = 10 (Fig. 4). «Here it is noticed that the curve first rises asymptotically and after only one period practically reaches the final steady state. This steady state is characterized by a very marked departure from the sinusoidal form».

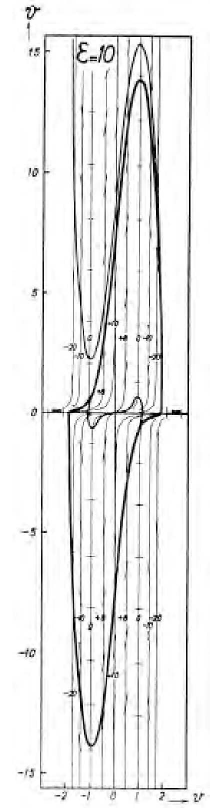

Fig. 4

Van der Pol points out that the oscillation contains numerous higher large-amplitude harmonics and demonstrates that the period T of the oscillation, instead of being equal to 2π (as in the case in which $\varepsilon << 1$), is approximately equal to ε and thus to RC. He calls this the *time of a relaxation*, from which derives the term *relaxation-oscillation* suggested for this type of phenomenon.

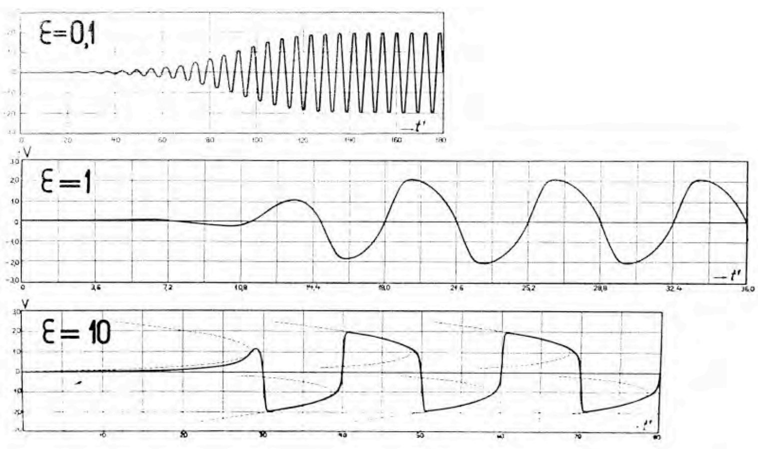

Fig. 5

A conventional type representation is thus provided for the solution in all three cases (Fig. 5). It should be noted how van der Pol's analysis is influenced by the specific modalities in the individual numeric cases as well as by the empirical phenomena they reflect. In the following part of the article, he actually refers to a series of devices or phenomena that could be used as examples of relaxation-oscillations. The definition could apply to Abraham-Bloch's multi-vibrator, an electric system consisting of two triodes, resistances and capacitators, that produces numerous high harmonics as it oscillates. Other examples cited towards the end of the article are «the well-known vibration of a neon-tube connected to a resistance and condenser in shunt», and «perhaps also heart-beats». This is the first reference made to the possibility of using relaxation-oscillations to model the heartbeat.

From the foregoing it is clear that this work opened up a dual line of development: on the one hand, a more detailed investigation of the general problem of non-linear oscillations, for the purpose of working out a general theory based on rigorous and standardized mathematical methods. On the other, the extension of the field of applications of the results obtained. Van der Pol was quite aware of this second avenue of developments and at the end of his article he endeavours to provide fresh examples, in addition to that of the triode, when he suggests the heartbeat analogy. This was actually a move in the direction of modelling, in which he obtained a substantial success by virtue of the elaboration of the heartbeat model and with the suggestion of other interesting analogies. We shall deal with this issue in section 4. On the other hand, as far as the development of a general theory of non-linear oscillations is concerned, van der Pol's contributions were affected by his limited mathematical background and, above all, by the fact that he was unaware of the contributions made by Poincaré and Lyapunoff to the theory of non-linear differential equations. It was above all the Soviet school of mathematics that took up the issues he raised and developed them in a significant way. We shall examine this aspect in a subsequent section. However, before doing so we must mention another important contribution made by van der Pol.

The contribution we are referring to is related to the van der Pol equation with a forcing term. It is generally considered to be related to work he did in 1927[8] and that appears to be a development of the 1926 paper described earlier. In fact, it consists of an English translation of a paper published in Dutch in 1924[9]. This could explain the delay with which it became known in international scientific literature.

As a matter of fact, in the 1924 paper we find the triode oscillator equation in its definitive form, albeit in the case in which it is subjected to a forcing term (namely a signal in the form of a continuous wave). To be precise, the equation is:

$$\ddot{x} - (\alpha - 3\gamma x^2)\dot{x} + \omega_0^2 x = B\omega_1^2 \sin\omega_1 t \tag{4}$$

It is no coincidence that van der Pol mentions only *en passant* the case of zero external electromotive force – a case that will be dealt with in the later 1926 paper (itself earlier than the unmodified English translation of this article). He dwells at length on the forcing case. As M. L. Cartwright[10] has pointed out, the method used by van der Pol to study the equation is particularly important, as it was adopted by the Soviet researchers who gave it a rigorous general re-formulation. This consisted in assuming that the form of the solution would be $x = a\sin\omega_1 t + b\cos\omega_1 t$, where a and b are functions of the time t that vary slowly. In this way differential equations were obtained for a and b , $\dot{a} = A(a,b)$, $\dot{b} = B(a,b)$, in which the second member functions are polynomials.

This is an important article also as far as the specific results are concerned. After pointing out that, in the absence of any forced term, the free frequency oscillation ω_0 has an amplitude of a_0

[8] Pol B. L. (van der) 1927.
[9] Pol B. L. (van der) 1924.
[10] Cartwright M. L. 1960.

obtained from the expression $a_0^2 = \frac{\alpha}{3/4\gamma}$,Van der Pol went on to say that near the resonance region ($\omega_1 \cong \omega_0$) free oscillations are suppressed by forced oscillations. Far from the resonance region, forced oscillation amplitude decreases below a_0. On continuing to desynchronize the system, the free oscillations begin to be set up with a small amplitude and a small correction of the frequency towards the frequency of forced oscillations. The upshot is that if amplitude is constant, only the forced oscillation is present. If in addition to the constant component of the amplitude a slow and periodic variation also occurs in it, then coexistence between free and forced oscillations occurs. «These oscillations - van der Pol remarks - react on one another due to the non-linear term containing γ in the equation, as opposed to cases of linear form in which this interaction is absent».

The study of the theory of synchronization of frequency specific to a non-linear oscillator with that of the external forcing led van der Pol describe the phenomenon of the demultiplication of frequency in the case of relaxation-oscillations. It should be noted that also this phenomenon was described by Poincaré in chapter XXVIII of the third volume of *Méthodes nouvelles de la mécanique céléste*, as Kryloff and Bogoliuboff pointed out in their comment to van der Pol's works. Once again, however, the issue was being tackled in a new context, above all from the technological standpoint, and it is interesting to note that van der Pol's empirical analysis succeeds in detecting the presence of a phenomenon that today goes by the name of "deterministic chaos" and the significance of which escaped not only van der Pol but even more so his contemporaries.

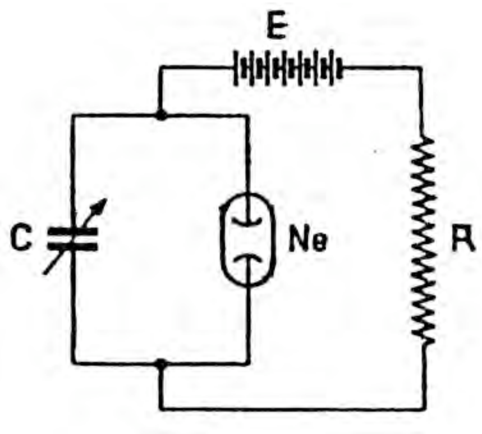

Fig. 6

A paper written in collaboration with J. van der Mark[11] describes an electric circuit which produces oscillations with relaxation (Fig. 6). In this case, *Ne* is a neon lamp. In the absence of an external electromotive force, the circuit oscillates with a period of $T = \alpha CR$, where α is a number of the order of one. If an external electromotive force is introduced, it is found that the system is capable of oscillating only at *discrete frequencies* determined by *integers sub-multiples of the external frequency*.

This is how the phenomenon is described in a later paper[12]: «… we apply a small external periodic force having the same oscillation frequency as the system. By subsequently gradually reducing the period specific to the system's relaxation, we find that it continues to oscillate with the same period as the external force, so that the system automatically becomes synchronized with the external forces. The system's frequency can be decreased practically to the lower octave without preventing it from oscillating in perfect synchrony with the external force. If we continue to reduce

[11] Pol B. L. (van der), Mark J. (van der) 1927.
[12] Pol B. L. (van der), Mark J. (van der) 1928b.

the system's period, its frequency abruptly rises to a value that is exactly *half* the frequency of the external force and the system is automatically maintained at this new frequency over an external range. A slight reduction in the frequency specific to the system, again without modifying the frequency of the external force, abruptly makes the system oscillate on the third *subharmonic* of the applied force, where it is again maintained over an extended range, etc.… We have been able to prolong this experience of *demultiplying frequency* up to a ratio of 200 : 1.»

In van der Pol's and van der Mark's experiment, the frequency of the oscillations was in each case determined by means of a telephone coupled to the system. They pointed out that «often an irregular noise is heard in the telephone receivers before the frequency jumps to the next lower value.» This observation has always escaped the attention of readers of the article. It clearly shows that van der Pol and van der Mark had come up against a phenomenon that may be interpreted in terms of the notion of "deterministic chaos". As is natural, the importance of the phenomenon they had detected escaped them. They actually wrote: «However, this is a subsidiary phenomenon, the main effect being the regular frequency demultiplication». Likewise, none of those who read the article at the time paid much attention to the observation. Even in more recent times, when Guckenheimer and Holmes dedicated their book[13] to van der Pol, defining him «a pioneer in a chaotic land», they made no specific reference to this result.

Fig. 7[14] shows a representation of the phenomenon of demultiplication of frequencies that gives rise to a discrete series of sub-harmonics. The dotted line indicates the frequency with which the system oscillates in the absence of the alternating electromotive force, while the hatched portions – as the authors say – "correspond to those settings of the condenser where an irregular noise is heard".

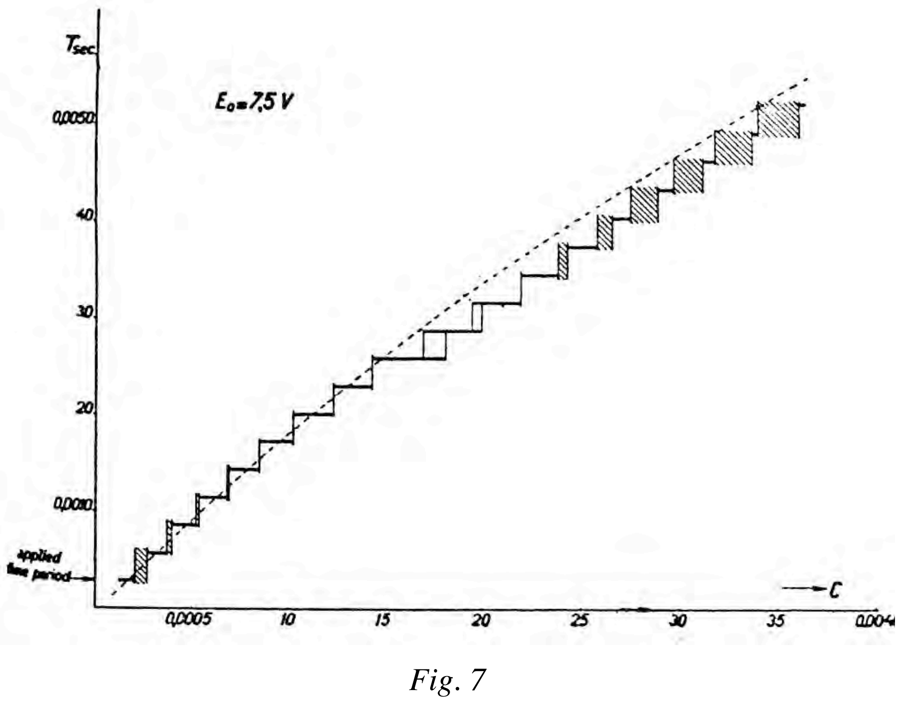

Fig. 7

In an unpublished letter written by van der Pol to the Italian mathematician Vito Volterra in 1930, he speaks of the phenomenon of the demultiplication of frequencies, suggesting that Volterra took it into account in the analysis of his population dynamics models. There is no record of any

[13] Guckenheimer J. Holmes P. 1983.

[14] From Pol B. L. (van der), Mark J. (van der) 1928b: in Pol B. L. (van der), Mark J. (van der) 1927 a very similar but less clear one appears.

reply from Volterra. It should be noted however that this was not the only occasion on which van der Pol showed an interest in Volterra's work on biomathematics. In a paper published in 1934 and dedicated to providing a general outline of the non-linear theory of electrical oscillations[15], van der Pol analysed the case of a triode oscillator with two degrees of freedom. In actual fact, this analysis was a representation of the results obtained by van der Pol thirteen years earlier[16]. Now, however, van der Pol was saying that the system of linear differential equations that he had derived in order to describe the interaction between the two oscillations was «exactly equal to those occurring in a now famous problem of parasitology, where the coexistence is investigated of two species, a host population and a parasite population. This problem was investigated by A. J. Lotka and by V. Volterra».

It is an established fact that the "Volterra-Lotka equations" were derived by Lotka and Volterra simultaneously in 1925: by the former in order to represent the dynamics between a population of parasites and one of hosts, by the latter to represent that between a population of preys and one of predators. Furthermore, Lotka had obtained the same equations in 1920 to describe the dynamics of an oscillating chemical reaction. A disagreeable controversy over priority ensued[17]. Van der Pol has the merit of not having entered the fray himself, although he would to some extent have been entitled to, having himself derived these equations in 1921. The only relevant observation that may be made is that obviously the development of non-linear analysis was in the nature of things and that the discovery of the simplest type of non-linear oscillator – that of Volterra-Lotka - was now to be expected in all contexts.

3. The contribution of the Soviet school

As we have seen, the concept of limit cycle was already known from the work of Poincaré and Lyapunoff, who had placed the problem of seeking periodic solutions for non-linear system problems on a rigorous footing. From this point of view, van der Pol's contribution does not represent any substantial progress and indeed involves the use of non rigorous, overly *ad hoc* analytical procedures. Nevertheless, in order to appreciate its importance, it must be pointed out that Poincaré and Lyapunoff's contributions had been confined to classical mechanics and mathematical physics, while the field of empirical problems, which inspired van der Pol, raises the question of equations that are somewhat different from those that he was accustomed to dealing with. This also raised the issue of adapting and generalizing Poincaré's and Lyapunoff's methods, which were considered inadequate for dealing with this new type of equation.

This is the framework within which the contribution by the Soviet mathematical school was working: it did not have so much the merit of inaugurating the treatment of this new applications problem, as to have reappraised, reordered and generalized it at the mathematical level in the light of Poincaré's and Lyapunoff's theory, after it too had been suitably re-elaborated. In this connection, it would seem that the peculiar value of van der Pol's contribution consisted in having ventured for the first time into a field that had hitherto attracted little attention. The Soviet mathematical school had the merit of having set the new problems in a suitable mathematical context.

One of the first articles acknowledging the relative importance of the new research was a

15 Pol B. L. (van der) 1934.
16 Pol B. L. (van der) 1921, 1922.
17 In this connection, see Israel G. 1982, 1988.

short note written in 1929 by A. Andronow[18]. In it the author points out that «the oscillations referred to as 'auto-entretenues' have for some years been arousing increasing interest in many fields of the natural sciences. These oscillations are governed by differential equations that differ from the ones studied in mathematical physics and classical mechanics. The systems in which these phenomena are produced are non conservative and maintain their oscillations by receiving energy from non periodic sources». Among the phenomena to which Andronow refers, a special place is occupied by the triode oscillator, as well as periodic reactions in chemistry and biology. It should be noted that the study of these phenomena have as their point of reference the work of van der Pol, Lotka and Volterra[19]. The main purpose of the note is to show that the self-oscillations arising out of a system characterized by simultaneous different equations of the type

$$\frac{dx}{dt} = P(x, y) \quad , \quad \frac{dy}{dt} = Q(x, y)$$

correspond to Poincaré's limit cycles. It goes no further, but it is intended in this way to show that «the theory of self-oscillations, in which almost exclusive use was hitherto made of non rigorous methods, is thus given, at least in the simplest case, a rigorous mathematical basis».

It should be pointed out that, in the meantime, the proof of the existence of a single period solution having the nature of a limit cycle of van der Pol's equation had been provided by A. Liénard[20]. Commenting and further developing his result, B. Decaux and Ph. Le Corbeiller pointed out that, by applying Poincaré's theory, it had thus been possible to obtain «a complete, mathematical and physical, explanation of the observed phenomenon»[21].

One of the most comprehensive contributions to the problems raised in radio engineering is contained in three scientific notes by N. Kryloff and N. Bogoliuboff, all published 1932[22], and the content of which was then summarized in a subsequent article[23], in which the authors also presented a historical outline of the topic and made some considerations concerning the contributions made by the various national schools to the new branch of learning they called "non-linear mechanics".

Kryloff and Bogoliuboff began from the consideration that the time had now come to devote greater attention to non-linear oscillations, which presented a number of unsolved mathematical difficulties, rather than continue to remain anchored to the now well-known theory of linear oscillations. The greatest of these difficulties consisted in the fact that in non-linear radio circuits oscillations may be produced in which the frequencies are linear combinations of the principal frequencies, so that the mathematical functions representing them have a quasi-periodic nature. Here we see the limits of Poincaré's classic methods. Indeed the development of quasi-periodic functions following the exponents of the parameters does not provide an adequate representation and above all does not converges uniformly on the whole real axis. The original pathway indicated by Kryloff and Bogoliuboff consists in seeking the development of the amplitudes, phases and frequencies directly rather than that of the function. The case they treat is precisely that of the van der Pol equation, for which they re-obtain in a rigorous way a series of results proved by van der Pol. The general nature of the approach nevertheless leads them to conclude that «the method by which we arrived at the results contained in these Notes are applicable and effective, as we were able to verify in the course of our research, in many other questions (for example, oscillations of synchronous machines, longitudinal stability of aircraft, etc.) and Chapters of modern physical

18 Andronow A. 1929.
19 For the latter two, see Israel G. 1993, 1988.
20 Liénard A. 1928.
21 Decaux B., Le Corbeiller P. 1931. See also Le Corbeiller P. 1932.
22 Kryloff N., Bogoliuboff N. 1932a, b, c.
23 Kryloff N., Bogoliuboff N. 1933.

mathematics (Quantum Mechanics), and can open the way, we think, to the creation of non-linear general mechanics».

The foundation of this branch was presented as a now achieved objective in the previously mentioned review article by the two authors published in 1933. After extensively underlining the fundamental importance of Poincaré's and Lyapunoff's work in the direction of working out rigorous methods for finding periodic solutions to non-linear differential equations, Kryloff and Bogoliuboff acknowledged van der Pol as having the merit of being the first «to draw the attention of the world of science to the need to develop special methods for treating non-linear problems in radio engineering, expressing himself in one of his papers as follows: "It is therefore somewhat surprising that up to the present, though several theoretical contributions to the problem have already appeared, the phenomenon has, as far as we are aware, only been dealt with in a linear theory"». The limitation of van der Pol's work nevertheless lies in the fact that he did not make use of Poincaré-Lyapunoff's methods, but rather of «ingenious procedures, nevertheless lacking in the necessary mathematical rigour». However, they also admitted that even the application of classical methods would not have been sufficient: «It is only right to observe that the comparatively non rigorous procedures of the distinguished Dutch scientist and applied by him *ad hoc* nevertheless give several indications concerning the nature of quasi-periodic oscillations, and Poincaré-Lyapunoff's methods, in their present state, in no way seem applicable to the study of these objects».

Credit for having drawn attention to the need to deal with radio engineering problems with the required mathematical rigour was given to Liénard and Cartan in France and to the school of Mandelstam and Papalexi in the Soviet Union, and mention was made of the results they obtained, as well those by Andronow and Witt. It was nevertheless pointed out that the problem remained of suitably re-elaborating the Poincaré-Lyapunoff theory in such a way as to adapt it to the treatment of quasi-periodic solutions of non-linear differential equations with self-oscillations. The results obtained in this direction seemed so solid as to justify the talk of a new branch of research to be denoted as "non-linear mechanics" and that contains a general study of non-linear oscillations regardless of the area in which they appear: astronomy, radio engineering, aerodynamics, chemical dynamics, animal population dynamics.

It should be noted that the terminology adopted had an obviously traditionalistic air about it. It might be claimed that it was too early at the time to use terms such as "models" or "modelling", even though this terminology was already being used, for instance, by von Neumann. In any case, reference to the term "mechanics" tended to include the new applications under a general heading in the classical mechanics field, rather than define a new area characterized by reference to a complex of mathematical structures. For the Soviet researchers, the emerging effectiveness of the new methods of non-linear analysis does not seem to have led to the introduction of a new relationship between mathematics and empirical phenomena, but rather to the extension of the domain occupied by the classical mechanics approach.

Also in this respect the "less rigorous" van der Pol intervenes by playing an important role of stimulus and innovation. The way in which he proposes the use of self-oscillations (and, in particular, relaxation oscillations) in studying a vast range of "new" phenomena is also related to the idea of the mathematical model and to the practice of modelling, without any compulsory references to the schemata of mechanics, and thus taking a significant step outside mechanistic reductionism.

4. The heartbeat "model"

In an article they wrote together in 1928[24], van der Pol and van der Mark drew up a list of phenomena that, in their opinion, could be gathered together under the heading of relaxation oscillations:

«Some instances of typical relaxation oscillations are: the æolian harp, a pneumatic hammer, the scratching noise of a knife on a plate, the waving of a flag in the wind, the humming noise sometimes made by a water-tap, the squeaking of a door, the multivibrator of Abraham and Bloch, the tetrode multivibrator, the periodic sparks obtained from a Wimshurst machine, the Wehnelt interruptor, the intermittent discharge of a condenser through a neon tube, the periodic re-occurrence of epidemics and of economic crises, the periodic density of an even number of species of animals living together, and the one species serving as a food for the other, the sleeping of flowers, the periodic reoccurrence of showers behind a depression, the shivering from cold, menstruation, and, finally, the beating of the heart. In all these examples the frequency of these periodic phenomena is not determined by the product of an elasticity and a mass but by some form of relaxation time»[25].

This list is a surprising one for many reasons. In the first place because of the heterogeneous nature of the phenomena cited, which all refer to completely different contexts. These phenomena are linked together only by an *analogy* consisting in the fact that they can all be described by a single mathematical *model*. The second reason is related to the vaguely intuitive and analogical nature of the reasons why the phenomena in question may be considered as self-oscillating. On the list of "examples of typical relaxation oscillations"only a few (electrical circuit, neon tube, and the other electrical equipment) may be considered self-oscillatory phenomena on solid theoretical and experimental grounds. In all other cases, the existence of relaxation self-oscillation and a limit cycle rests solely on a vague, non rigorous intuition. The idea of representing non physical natural phenomena described on the list by means of a "van der Pol oscillator" was born and died in this article, with one single exception: the heartbeat. In this article we actually find the description of a mathematical model that became the famous prototype of a long series of mathematical models of the heartbeat which proved valid and to be of effective practical utility. The other proposals by van der Pol practically all remained a dead letter. When van der Pol speaks of the possibility of describing in mathematical terms coexistence among animal species he mentions Volterra's results. But also this reference is somewhat generic (as was that of the Soviet scientists), because Volterra's classical models have no limit cycle. The idea of describing the dynamics of economic crises by means of self-oscillations has had a rather modest outcome.

And yet, precisely this open-mindedness in seeking analogies among different phenomena, trying in this way to bring them all together within the framework of a single mathematical representation is one of the first very clear manifestations of the *modelling* approach. For this reason, we consider this passage to be of considerable historical interest as it expresses a clear-cut

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[24] Pol B. L. (van der), Mark J. (van der) 1928b.

[25] It may be of interest to compare this passage with the similar one contained in Pol B. L. (van der), Mark J. (van der) 1928b. Here, as well as the addition of other examples, there is another, clearer description of the analogies concerned. He thus remarks: "Let us take, for example, the aeolian harp, which consists of a taut cord against which the wind plays. A detailed examination shows that behind the cord vortexes are formed to the right and the left which are propagated away from the harp, thus making room for new vortexes that are created and that move away in the same way. Thus in the case of an aeolian harp, as in that of the wind blowing across telegraphic wires to produce a whistling sound, so the period of the sound produced is determined by a relaxation time and has nothing to do with the natural period of the cord which vibrates in a sinusoidal manner". We have here a very clear description of the socalled Bénard-von Kármán phenomenon.

distancing from the conventional paradigms of classical reductionism[26].

It has been said that it is only in the case of the heartbeat that van der Pol's analogy proved truly felicitous. The way this model was constructed is quite surprising and it is interesting to follow a brief description of it.

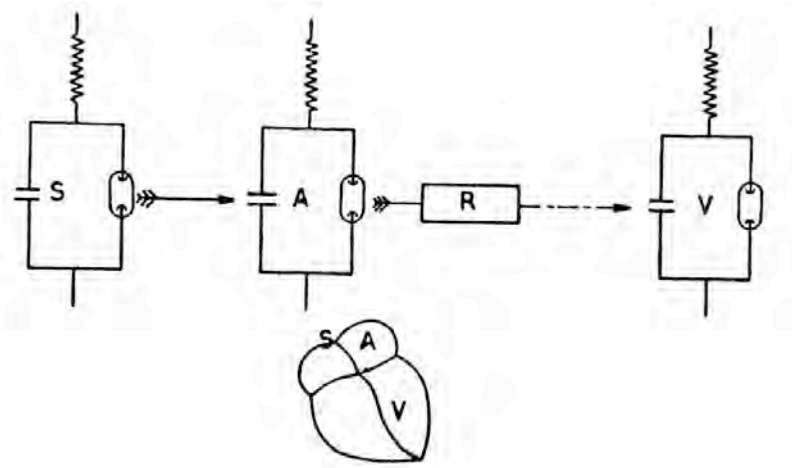


Fig. 8

For a long time it was thought that the heart's beating was regulated via the central nervous system. However, the development of modern physiology showed that this was not at all the case: although the nervous system can affect the cardiac rhythm it plays no actual 'pace-making' role in this process. Although the heart possesses the property of *irritability* which makes it sensitive to stimuli from the nervous system, it is at the origin of its own *contractility*. Furthermore, each part of the heart is capable of generating this contraction autonomously. The different parts of the heart that have the function of stimulating contraction are nevertheless arranged in a precise hierarchical scheme. The role of "pace-maker" is played by a system of cells known as the *sinus node* (*S* in Figure 8). The beating of a healthy heart is regulated by the stimulus from the sinus node and is propagated through its various parts, starting from the atria *(A)*. If the sinus node stimulus fails to reach the atria, they contract with a rhythm of their own which is slower than that of the sinus node. The same thing happens in the other parts of the heart. The sinus node stimulus is transmitted from the atria to the ventricles *(V)* through a bundle of muscle fibres known as the *atrioventricular bundle* or *bundle of His* (*H*). In this way, you get a rough outline of a hierarchic breakdown of the heart into five parts: more detailed breakdown may lead to more precise descriptions. The hierarchy adopted by van der Pol is the simplest possible. It is based on the hypothesis that the contractions of the atria and the ventricles are synchronous (which is only very approximately true) and schematically represents the heart as a system with three degrees of freedom: the sinus node (*S*), the atrium (*A*), the ventricle (*V*), with the hierarchy $S \rightarrow A \rightarrow (H) \rightarrow V$ (in which the passage $A \rightarrow V$ is not a direct one but takes place through the mediation of the bundle of His *H*).

Van der Pol's idea was thus to liken the heart to an electrical system with three degrees of freedom — the three components *S, A, V* — each functioning like an electrical system producing relaxation oscillations. The electrical system capable of producing relaxation oscillations, and chosen by van der Pol as his model, is a circuit system of the type represented in Fig. 6. Each circuit is made up of a neon lamp *Ne*, a condenser *C* with a capacitance of about 1 microfarad, a resistance *R* of 1 megaohm and a battery *E* of 180 volts. Since the condenser charging time in seconds is given by the product of the capacitance *C* (in farads) and the resistance *R* (in ohms), the period *T* of the relaxation oscillation is $T_{rel} \cong CR = 10^{-6} \cdot 10^{6} = 1$ second. The neon lamp will thus produce one

[26] This question has been discussed in detail in Israel G. 1996.

flash approximately every second.

Van der Pol's electrical heart model is thus constructed by hierarchically coupling three of these systems: the first of these represents the sinus node, the second the atrium and the third the ventricle (Figure 8). Transmission of the stimulus from *A* to *V* is represented by means of a retarding system achieved by means of a fourth circuit *R* containing a neon tube and, in van der Pol's words, «imitating the finite time taken for a stimulus to be transmitted from the atrium through the atrio-ventricular bundle to the ventricle».

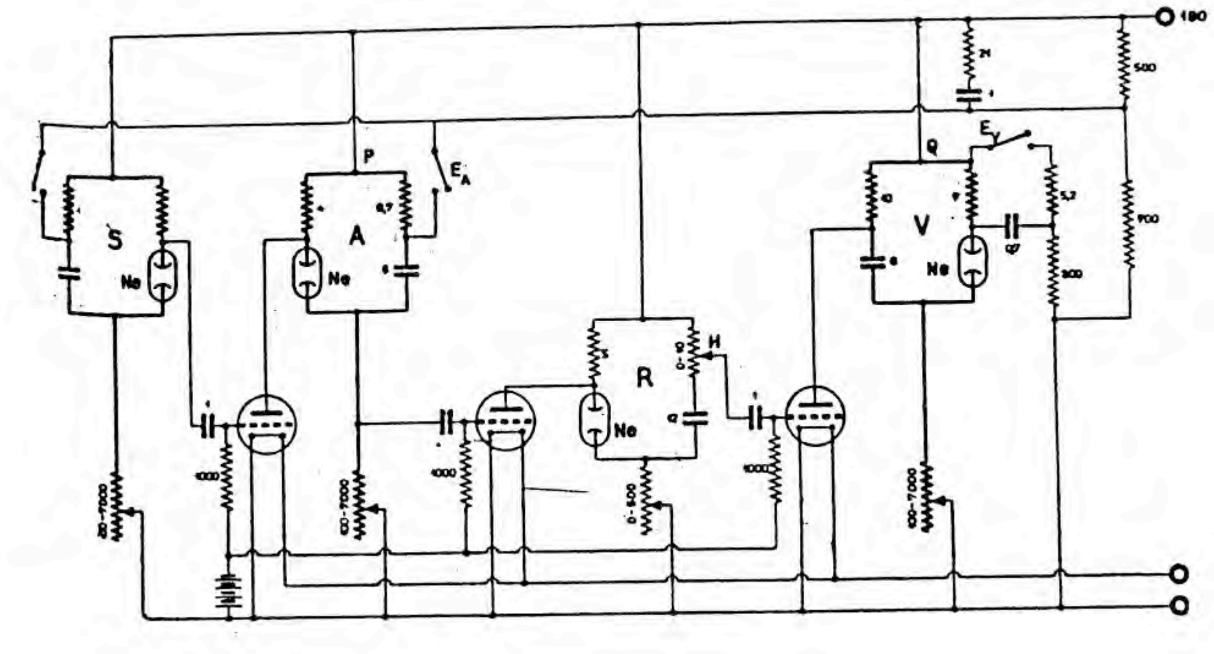


Fig. 9

The general electrical diagram of the model is shown in Figure 9. Each flash of the system corresponds to the activity of one part of the "heart". The system *S* produces relaxation oscillations. By means of the first triode, this stimulus is unidirectionally transmitted to the second relaxation system *A* (the atrium). The stimulation of the atrium is unidirectionally transmitted from the second triode to the retarding system *R* and from here to the third triode, to the ventricle *V*. The instrument was equipped with three keys by means of which a short electrical impulse could be transmitted to the three systems *S*, *A*, *V*, so as to simulate stimuli starting from different points from the usual ones and thus capable of producing an extra systole of the sinus node, the atrium or the ventricle. The *R* system is used to simulate lesions to the bundle of His, the socalled "atrio-ventricular blocks" («the coupling between *A* and *V* (the auricle and ventricle) can be varied at will, thus imitating the beautiful experiments of Erlanger of gradually clamping the bundle of His» — were the observations made by van der Pol and van der Mark).

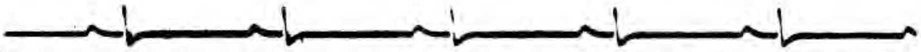
As can be seen in Figure 9, each relaxation system consists of a variable resistance coupled in series with a neon tube and a fixed condenser. A discharge from the latter through the neon tube represents a contraction by one of the parts of the heart. The frequency of each of the systems is regulated by the variable resistance. Two adjacent relaxation systems are coupled by means of a triode in the following way. Let us consider the sinus node-atrium coupling. One of the extremities is connected to the triode grid by means of a condenser and a grid resistance. The plate current of this triode must pass through a resistance situated above the neon tube of the atrium. In the instant in which the neon tube is illuminated, a current passes through its resistance. The potential at the lower extremity thus becomes more negative, which causes a reduction in the potential of the grid connected to this extremity and thus a reduction in the current plate of the triode. The potential difference at the extremities of its anodic resistance thus decreases, which leads to an increase in the potential difference at the extremities of the neon tube of the atrium. If at this instant the potential difference at the extremities of the tube is not too far from its discharge potential, the tube is illuminated. We then see the two tubes coming on together. If the potential difference is not high enough, seeing that the atrium condenser is still not sufficiently charged, the discharge potential is not attained and nothing happens, since the excitation occurs during the "refractory" period. We thus have an atrioventricular block.

The retarding system is also a relaxation system, although the current in it is too strong to produce oscillations. A continuous current passes through the neon tube and the system is overloaded, so that no periodic phenomenon can occur. Nevertheless, if the potential difference at the extremities of the neon tube is reduced, the luminous discharge is interrupted, but the condenser is recharged and when the discharge potential is attained the tube again produces a flash. The time elapsing between one extinction and the next discharge is the transmission delay and is equal to the relaxation time RC. External excitation thus produces only one beat. The retardation system is coupled to the ventricle in the manner described above. Since the inserted resistance is variable, ventricle excitation may be varied, as though it were constraining the bundle of His.

In Figure 9 we see that each relaxation system is provided with another resistance, like the one that, in the sinus node system, connects the condenser to the upper extremity of the high tension source. A key on the back of the apparatus allows to discharge a condenser through this resistance and to generate a stimulus in the relaxation system. Other keys can be used to generate other types of additional stimuli (or "extra systoles").

We shall skip describing the way "electrocardiograms" are made (i.e. current trend plots) by means of a special filter circuit and recorded with an oscillograph, as they are merely electrical devices that add nothing to the conceptual basis of the model.

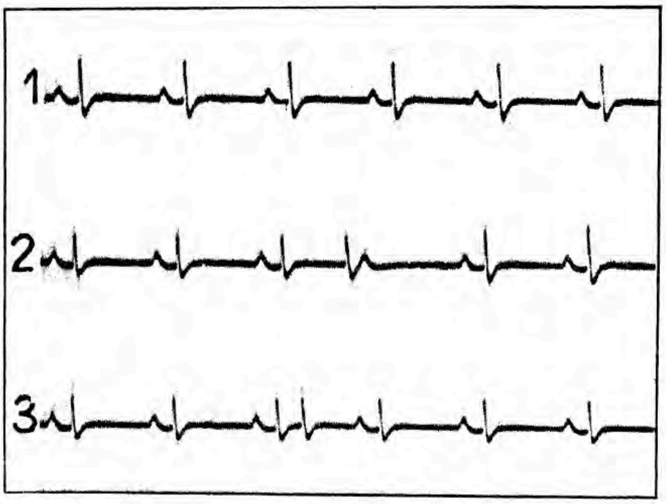
We shall rather have a direct look at what we obtain by making these measurements, that is, the type of electrocardiograms obtained with the artificial heart and the comparisons it is possible to make with the behaviour of the human heart.



Typical electrocardiogram of the artificial heart. The P top and the QRS complex are clearly visible. The T top however is missing, due to insufficient definite data at hand as regards its origin.

Fig. 10

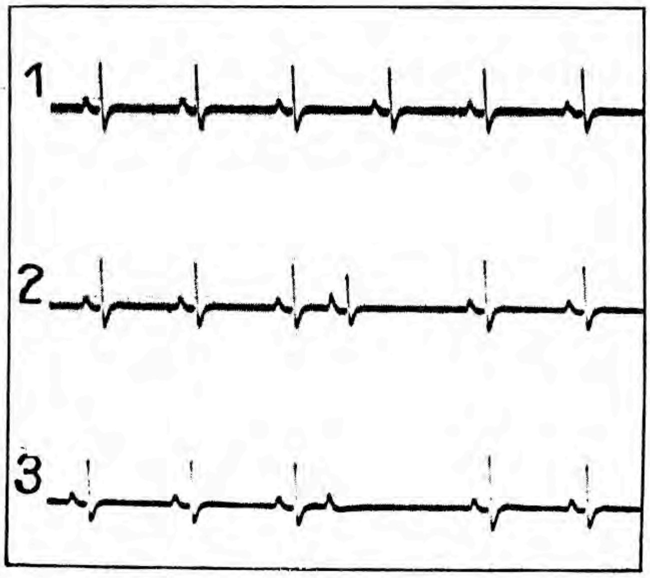
The result is represented in Figure 10. The analogy with the electrocardiogram of the human heart is quite apparent. We find the main components of the latter (the P wave and the QRS complex) with the exception of the T wave. In any case, the significance of the latter wave was not fully clear at the time: «as the origin of the T top in the electrocardiogram of the human heart is not quite certain yet, we could not insert a representing mechanism for it».



Ventricular extrasystolæ :—1, normal heart beat; 2, late ventricular extrasystole resulting in the ventricle being in the refractory state when the next following normal stimulus arrives from the auricle; 3, early ventricular extrasystole; here the ventricle is *not* any more in the refractory period when the next following normal stimulus arrives from the auricle and thus an *interpolated ventricular* systole is obtained.

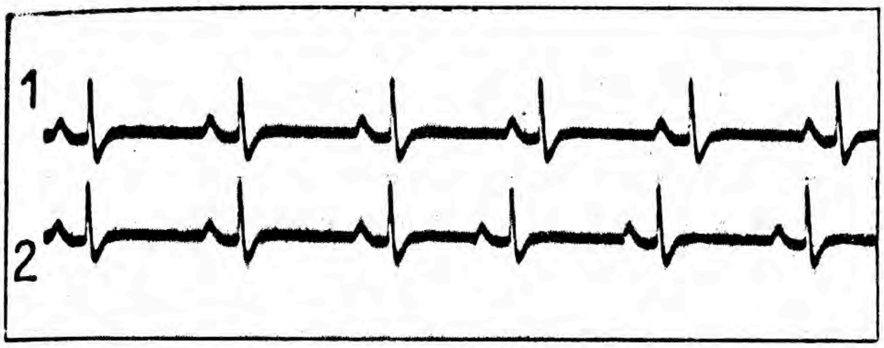
Fig. 11

In their work, van der Pol and van der Mark demonstrated that, by transmitting impulses to the electric heart through the keys located at the back of the apparatus, it was possible to simulate the three standard types of human heart extra systole: ventricular (Fig. 11), atrial (Fig. 12) and sinus extra systoles (Fig. 13).



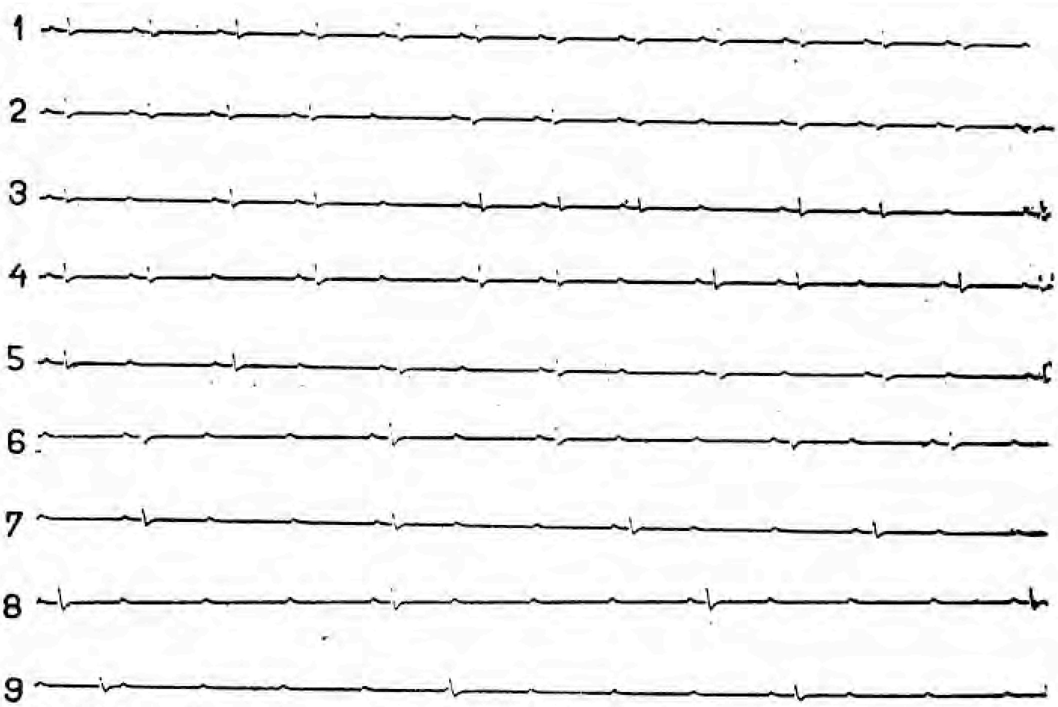
1. **Normal heart beat.**
2. **Auricular extrasystole (with the ventricle responding).**
3. **Auricular extrasystole (ventricle still in refractory period).**

Fig. 12



1. Normal heart beat.
2. Sinus extrasystole disturbing the whole heart rhythm.

Fig. 13



Electrocardiograms from the artificial heart obtained by gradually reducing the coupling between the A and V system (clamping the bundle of His). The development of 2 : 1, 3 : 1, and 4 : 1, as well as complete heart block, is clearly shown.

Fig. 14

Van der Pol and Van der Mark also managed to simulate lesions to the bundle of His, which in the human heart entrained the loss of part or all of the stimuli coming from the sinus and, as a result, of the ventricular beats. This loss may be an occasional event, although severe damage can lead to a systematic loss of ventricular beats. The ventricular beats may then occur at the rate of one beat every two beats of the sinus or one beat every three beats of the sinus, and so on, up to a state of complete dissociation. In the latter case, the ventricle beats at its own rate ("idioventricular rhythm”). This kind of situation has long been known under the name of *atrioventricular blocks*; in particular, in cases of a well-defined ratio between beats, they are known as *type n:1 Wenckebach blocks* (where *n* is the number of sinus beats which may vary from 2 to 4). van der Pol’s model allows a complete classification of these blocks to be obtained. Figure 14 shows the results obtained. This classification was not completely known at the time of publication of van der Pol’s and van der Mark’s article. The model could thus be used only to predict and classify cardiac disorders that were only partially understood at the time.

It should be noted that the heartbeat model cannot be considered a “pure” mathematical model in the modern sense of the term. Van der Pol’s equation is not “the” model of the heartbeat, but only the fundamental component of its interpretation. Modelling the heart entails the (also material) construction of a physical simulation apparatus or electrical circuit. Nevertheless, the equation of this circuit is not provided, and would have to be obtained from a system of interdependent van der Pol equations. It would be a system of such complexity as to defy not only the analytical resources deployed by van der Pol, but even those available today.

In any case, even via the experimental physical mediation of apparatus designed to compare the theoretical model with reality, van der Pol’s heartbeat model may certainly be considered as the prototype of a form of mathematical modelling which displays two of its characteristic features: the method of *mathematical analogy*, and the special role assigned to *non-linearity*.

5. Concluding remarks

The majority of the historical reconstructions of the developments in non-linear analysis during the 20th century followed the following stereotype. Towards the end of the previous century, important progress had been accomplished in this field by Poincaré, and later Lyapunoff. These results had very little impact and practically fell into oblivion, particularly in western science, with the exception of a few isolated developments, mainly by George Birkhoff. On the other hand, they were taken up energetically by Soviet mathematical physics which, starting in the ‘thirties, also as a result of the stimulation from a number of innovative technological applications, made an outstanding contribution to non-linear analysis. This Soviet contribution was fully acknowledged by western science only after World War II.

Although such a reconstruction contains several correct elements, a number of substantial corrections are needed.

There is no questioning the extremely important contribution Poincaré made to non-linear analysis towards the end of the nineteenth century and in particular to the problem of finding periodic solutions for non-linear systems. It must however be borne in mind that the context in which this research was carried out was that of classical mechanics. Indeed, - as illustrated elsewhere[27] - it is incorporated in a programme for the development of analysis and in a conception of the relationship between analysis, physics and geometry that may well be described as “conservative” or at least concerned with defending the strongholds of mathematical-physical reductionism. Nevertheless, at the beginning of the twentieth century, the attention of the scientific world is directed elsewhere. On the one hand, we have the developments in new physics: the theory of relativity and quantum mechanics. On the other, many scientists, particularly in the field of applied science and engineering, are exploring the implications of the new technologies: among these, radio engineering played a major role and the example of van der Pol is typical in this respect. To this situation must be added the impetuous development of mathematization in the field of the economic and biological sciences which is opening up hitherto undreamt of prospects and

[27] Israel G., Menghini M. 1998.

topics[28]. It is thus not surprising that interest waned in classical mechanics topics and that Poincaré's work in this field aroused less attention than it deserved, even to the extent of obscuring the value it had precisely in the development of that non-linear analysis which was of such great importance in the study of the new problems.

However, it would not be correct to consider this state of affairs as a regression in western science and to attribute to Soviet science all the merit of playing a propulsive role in taking up and continuing the work of Poincaré. It should indeed be noted that the interest shown by the Soviet school in Poincaré's work was precisely a consequence of his attachment to the classical topics of mechanics and his substantial remoteness from many of the new developments in physics. It would be necessary to make a careful study of the role played by radio engineering research also in the Soviet Union in the development of non-linear analysis, although there is evidence to believe that it did not reach the high level it reached in the West, in particular in an environment such as the Philips company. Moreover, we have seen that the Soviet scientists themselves acknowledged the decisive importance of van der Pol's work, despite the fact that the mathematical methods he used were not particularly rigorous. A necessary step was to acknowledge that Poincaré's methods, as formulated in the context of celestial mechanics, were inadequate as a basis for tackling the study of the mathematical structures proposed by radio engineering. This in no way diminishes the importance of the contribution made by the Soviet scientists; they put research on non-linear mathematics on the right track. What we mean is that, without taking into account the powerful impulse given by the applied research developed in the early decades of the century - however *ad hoc*, non rigorous and even rough and ready – it is impossible to understand why, starting from the 'thirties, there were such massive developments in non-linear analysis and even to understand why Poincaré's work was resumed and had such a great success.

New technology and new practical applications that stimulate the development of a new mathematics: this is the theme emerging from the history of the first few decades of the century and which challenges the view of a mathematics that is always in advance of the analysis of phenomena, a set of empty forms ready to be filled out with "possible content"[29].

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28 See Israel G. 2000.

29 Bourbaki N. 1948.

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Appendix

Two unpublished letters written by Balthasar van der Pol to Vito Volterra

DR. BALTH. VAN DER POL
JAN SMITZLAAN 12
EINDHOVEN
HOLLAND

30-12-1929

Monsieur le Professeur V. Volterra
ROME.
Cher Monsieur,

C'est déjà depuis quelques années que je m'occupe aux recherches générales theoretiques et experimentales sur des phénomènes periodiques dans la nature, et je prends la liberté de vous envoyer, ci-enclus, quelques tirages à part de mes articles sur ce sujet.

Je sais, que vous avez publié plusieurs recherches importantes sur les solutions periodiques d'équations differentielles et intégrales. Parce que il est très difficile pour moi de trouver ces publications complètes, je vous serais très reconnaissant, si vous voudriez bien m'envoyer des tirages à part de vos travaux sur ce sujet. Pour le cas que vous n'avez plus de tirages à part, vous m'obligerez beaucoup, si vous voudriez bien me dire, où je peux trouver vos publications précieuses.

Veuillez agréer, cher Monsieur, avec mes vives remerciments, l'expression de mes sentiments les plus distingués.

Balth. van der Pol.

NATUURKUNDIG LABORATORIUM
DER
N. V. PHILIPS'
GLOEILAMPENFABRIEKEN
KASTANJELAAN
TE EINDHOVEN

Eindhoven, le 28 janvier 1930
(Holland)

Monsieur le Sénateur
Professor Vito Volterra
Via in Lucina

ROMA.

Monsieur,

J'ai le plaisir de vous remercier beaucoup de l'aimable envoi de vos deux travaux, dont sûrtout l'article: "Variations and Fluctuations of the number of Individuals in animal species living together" a mon plus grand intérêt.

Permettez-moi de faire une petite observation, (faite avec l'intention de prédire des nouvelles possibilités futures de vos équations différentielles non-linéaires), en connection avec le cas, traité par vous sur pag. 49, où la "coefficience of increase" ε_r montre des petites fluctuations périodiques.

Nous avons constaté, pendant les dernières années théoriquement de même qu'expérimentellement, qu'une petite ride periodique avec une période ω_o, soit qu'elle parait dans un des coefficients d'une équation différentielle non-linéaire de deuxième ordre, soit qu'elle paraît comme terme libre dans le deuxième membre (comme force extérieure) peut avoir une influence fundamentelle sur le cours des phénomènes.

Aussi loin que, quand la période ω_1 avec laquelle le système oscillera, si les coefficients ε_r sont constants, est dans le voisinage de $\frac{1}{2}\omega_o$, $\frac{1}{3}\omega_o$, $\frac{1}{4}\omega_o$ ...., le système synchronisera automatiquement avec cette harmonique inférieure de ω_o (dites $\frac{\omega_o}{n}$) qui est le plus près de ω_1.

Dans le cas, traité par vous, où vous contemplez la vie ensemble de deux specimens, où l'un sert comme nourriture de l'autre, ceci reviendrait au suivant:

Quand l'oscillation libre du système avec des coefficients constants ε_r aurait une période de, par exemple $3\frac{1}{4}$ ans, alors, quand les ε_r's sont une petite ride avec une periode de precisément 1 an, le système total choisira automatiquement une période d'exactement 3 ans.

J'espère que cette observation vous intéresse, et si elle donnerait lieu à une observation sur ce sujet de votre part, il me serait très agréable de recevoir vos nouvelles.

Avec mes remerciments réitérés pour l'envoi de vos travaux si importants, veuillez agréer, Monsieur, l'expression de mes sentiments les plus sincères.

Dr. Balth. van der Pol.

P.S. J'ai encore le plaisir de vous faire parvenir par la présente un article: "Frequency Demultiplication" de M. van der Mark et moi, le sujet duquel est intimement lié à ce que nous avons traité dans cette lettre.

Annexe.